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Quantized Space-Time, Torsion, and Magnetic Monopoles

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Within the tetrad formalism we introduce quantized space-time in the curvilinear case by using general coordinate transformations with noncommuting terms. Fermion and boson fields are studied and the affine connection is also defined in this space. It is shown that space-time torsion and magnetic monopoles appear as consequences of the theory with quantized space-time at small distances. This method may open a new way of understanding topological structure of space-time.

1. INTRODUCTION

Concept of space-time and its properties with respect to matter structure plays an important role in developing the physical science. Space-time may be understood as an *a priori* base or theater in which physical phenomena take place and its interrelations with the material world are those of dialectical unity.

In classical works due to Newton, Einstein, and their followers the concept of space-time, and the interrelation between its structure and properties of matter, has received great understanding and further developments. Recently, this problem is very up-to-date and has received much attention with the successes of high-energy physics experiments which allow one to probe distances down to 10^{-16}–10^{-17} cm. In this connection, different possible structures of space-time at small distances are discussed intensively, which may become the natural arena of future physical theory. An introduction and general situation of this problem together with the related literature have been presented by Prugovečki (1984), Namsrai (1985a, 1985b), and therefore, we do not discuss it here again. We only notice that among the different proposed structure of space-time at small distances, quantized

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space-time structure or quantum geometry has become the focus of wide interest at present, owing to its role in two programs of representing gauge theories by random surfaces and strings (Polyakov, 1981; Gomez, 1982) and of the construction of the unified theory of elementary particle interactions including gravitation, based on the theory of strings and superstrings (Scherk and Schwarz, 1974; Schwarz, 1982; Green and Schwarz, 1984; Fradkin and Tseytlin, 1985). Here we will generalize results obtained in a previous paper (Namsrai, 1985b) in which we have discussed some possibilities of introducing quantized space-time at small distances and presented a concrete form its realization. In our scheme, it is proposed that there is no exact conceptual meaning of definite space-time points, i.e., the components of coordinates become operator valued and are not commutative:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] \neq 0 \qquad \text{for } \mu \neq \nu$$

where points $\hat{x}^{\mu}$ of quantized space-time consist of two parts:

$$x^{\mu} \Rightarrow \hat{x}^{\mu} = x^{\mu} + l\gamma^{\mu} \tag{1}$$

Here x^{μ} are usual c-number coordinates of nonquantized space-time, parameter l means a value of the fundamental length, and γ^{μ} are Dirac γ matrices. Within this simple realization of quantized space-time we have constructed quantum field theory free from ultraviolet divergences and have considered its consequences.

In the language of coordinate transformations formula (1) may be regarded as a global coordinate transformation, i.e., the parameter γ^{μ} does not depend on space-time points x^{μ}. The aim of this work is to generalize quantized flat space-time with coordinates (1) to curvilinear ones and to study geometrical aspect of this generic scheme. In this case instead of the global coordinate transformation (1) we shall consider the following general coordinate transformations:

$$x^{\mu} \Rightarrow \hat{x}^{\mu} = x^{\mu} + l\Gamma^{\mu}(x) \tag{2}$$

where $\Gamma^{\mu}(x)$ are arbitrary noncommutative functions of the points x^{μ}.

It turns out that reparameterizations of space-time by means of formula (2) lead to quantized space-time and give unexpected interesting results for physical applications. In particular, from the given model, it follows immediately that quantized space-time with the components (2) gives rise to the appearance of the space-time torsion and to existence of magnetic monopoles. Moreover, it opens a way to the extensions of general relativity formalism with the fundamental length.

In Section 2 we introduce quantized space-time with coordinates (2) and give the transition method to large-scale space-time for different physical fields. Sections 3 and 4 are devoted to the representation of the affine connection and torsion tensor for the quantized space-time, respectively. An interesting possibility of the existence of the magnetic monopole as a direct consequence of the quantized space-time is considered in Section 5.

2. SPACE-TIME WITH QUANTIZED COORDINATES

Recently, the theoretical and experimental successes of high-energy physics dictate a deeper level of understanding of the structure of space-time and its properties at small distances. However, our usual concepts of space-time are confirmed experimentally to be valid to distances of the order 10^{-15}-10^{-16} cm (see Namsrai, 1985b). This tells us that if different possible structure (discrete, stochastic, quantized, etc.) of space-time may exist at small distances, its appearance may be taken into account as a small background over the entire continuous space-time, or in other words, observable effects and contributions to them due to these expected structures to any physical processes, indeed, are very small. Thus, we assume that space-time with quantized coordinates is slightly different from classical continuous space-time, i.e., its coordinates may be formed by means of formula (2), where Γ^μ are arbitrary matrices with zero trace and, in general, depending on classical coordinates x^μ. From (2) it follows immediately that the commutator of quantized coordinates acquires the form

$$[\hat{x}^\mu, \hat{x}^\nu] = lG^{\mu\nu}$$

where

$$G^{\mu\nu} = x^\mu\Gamma^\nu - x^\nu\Gamma^\mu + l(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)$$

It is easily seen that $G^{\mu\nu}$ is antisymmetric over indices μ, ν and may be regarded as an operator in space-time with coordinates $\hat{x}^\mu$. In this sense, geometry of this space is similar to the geometry of Snyder (1947).

An important problem of the construction of the theory is how to pass from quantized space-time in the microworld to nonquantized one on a large scale. This procedure requires some mathematical method depending on the concrete realization of the introduction of quantized space-time into physics. We can distinguish two mathematical procedures:

(i) Limiting transition of the parameter of the theory, which characterizes discrete or quantized space-time. In our case, this parameter is the value of the fundamental length l and the passage to the large scale means that $l \to 0$.

(ii) The second way of introducing the method of quantized space-time may be regarded as a transformation of one coordinate system to another [like (2)] and that all observable physical processes may be understood as averaged values over the background of quantized space-time. In a given case, the averaging procedure is reduced to take trace of Γ^μ matrices, for example,

$$x^\mu = \langle \hat{x}^\mu \rangle = \frac{1}{d} \operatorname{Sp}(\hat{x}^\mu) \tag{3}$$

where d means the dimension of space-time, in particular, for Minkowski space $d = 4$.

Now we study fermion and boson fields in the flat quantized space-time (2). For simplicity, we assume $\Gamma^\mu = i\gamma^\mu\gamma^5$. For the fermion field we have

$$\psi(\hat{x}) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ip\hat{x}} u(p) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ipx} v(p)$$

where

$$v(p) = e^{\gamma^5 \hat{p} l} u(p) = \left[\cosh l(-p^2)^{1/2} + \frac{\gamma^5 \hat{p}}{(-p^2)^{1/2}} \sinh l(-p^2)^{1/2} \right] u(p)$$

Here $u(p)$ is the usual Dirac spinor, obeying the standard Dirac equation

$$(\hat{p} - m) u(p) = 0$$

In our case, the Green's function of the fermion field satisfies the following equation:

$$\bar{\psi}(\hat{x}') G_R^{-1}(\hat{x}' - \hat{x}) \psi(\hat{x}) = \delta^{(4)}(\hat{x}' - \hat{x})$$

or in the momentum space it takes the form

$$\bar{v}(p)(m - \hat{p}) v(p) = \bar{u}(p) G_R^{-1}(\hat{p}) u(p)$$

where

$$\bar{v}(p) = \bar{u}(p) \left[\cosh l(-p^2)^{1/2} + \frac{\gamma^5 \hat{p}}{(-p^2)^{1/2}} \sinh l(-p^2)^{1/2} \right]$$

In accordance with the averaging procedure discussed above (see, also Namsrai, 1985b) we have

$$G(\hat{p}) = \langle G_R(\hat{p}) \rangle = [m(p^2) - \hat{p} - i\varepsilon]^{-1} \tag{4}$$

where $m(p^2) = m \cosh(2l(-p^2)^{1/2})$.

Let us consider the vector boson field in the quantized space-time and its structure having the form

$$A_\mu(\hat{x}) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ip\hat{x}} e_\mu(p) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ipx} \varepsilon_\mu(p)$$

here

$$\varepsilon_\mu(p) = e^{\gamma^5 \hat{p} l}\, e_\mu(p) = \left[\cosh l(-p^2)^{1/2} + \frac{\gamma^5 \hat{p}}{(-p^2)^{1/2}} \sinh l(-p^2)^{1/2}\right] e_\mu(p)$$

and $e_\mu(p)$ is the usual polarization vector of the boson field, for which the following relation holds:

$$p_\mu\, e_\mu(p) = 0$$

For the generalized polarization vector $\varepsilon_\mu(p)$ obtained in accordance with the quantized property of space-time we have also

$$p_\mu \varepsilon_\mu(p) = 0$$

Therefore, spin structure of the vector boson particle is determined by the standard procedure as in quantum electrodynamics, and its Green's function takes the form (see, also Namsrai, 1985b)

$$\Delta_{\mu\nu}(p) \Rightarrow \mathscr{D}_{\mu\nu}(p) = -\frac{g_{\mu\nu}}{p^2} \frac{1}{\cosh 2l(-p^2)^{1/2}}$$

For the scalar boson field the causal Green's function acquires the following form in the quantized space-time with the coordinates $\hat{x}^\mu = x^\mu + il\gamma^\mu\gamma^5$:

$$\Delta(p^2) \Rightarrow \mathscr{D}(p^2) = \frac{1}{m^2 - p^2 - i\varepsilon} \cosh^{-1}[2l(-p^2)^{1/2}] \tag{5}$$

Thus, from formulas (4) and (5) we see that the obtained causal Green's functions for the fermion and boson fields coincide with the propagators of the corresponding extended (or spread-out) fields investigated in the nonlocal-stochastic quantum field theory due to Efimov (1977) and Namsrai (1985a). However, some slight difference appears in the fermion field case. In our scheme, a spread-out or nonlocal property of the fermion field is determined by means of its mass value only. If a fermion is massless particle then it is local always in quantized space-time. This fact is very important since one can consider the fermion as the source in interaction processes between elementary particles, while the boson field carrying the interaction signal between them becomes nonlocal, like a wave packet.

We notice that in accordance with (4) Einstein's relation $E = mc^2$ is changed in our case and takes the form $E = mc^2(1 + \frac{1}{2}m^2 l^2)$. Einstein's formula is valid up to energy (mass) values attained recently by the world better accelerators. From this it is easy to obtain the following restriction on the parameter of the theory $l \lesssim 10^{-16}$ cm (see, also Namsrai, 1985b).

It is interesting to note that for the fermion field case analogous results were obtained by Markov (1958) (see, for detail, Blokhintsev, 1973) in the stochastic theory based on the hypothesis of stochastic space-time with a small additional fluctuational term of the type of (1): $x^\nu \Rightarrow \hat{x}^\nu = x^\nu + a^\nu$, where a^ν is a random vector with some distribution. These authors attempted to link the inner structure of particles with this vector a^ν. Our earlier work (Namsrai, 1985b) belongs to this direction and as shown by us some concrete method of introducing quantized space-time leads to the nonlocal theory of quantized field theory (Efimov, 1977) constructed by means of an idea of stochasticity of space-time (Namsrai, 1985a). In general, we expect that the two ideas, stochastic and quantized structures of space-time at small distances, are very close to each other, at least at the mathematical method level.

Now we turn to the problem of how to realize the quantized space-time idea by means of transformation language of coordinate systems.

3. THE AFFINE CONNECTION

The affine connection is the more convenient method for the study of physical processes taking place in different coordinate systems and their covariant description under mutual transformations between them. The affine connection method is based on the principle of the universal constant velocity of light, expressed by the concept of the square interval or proper time of events. The essence of the affine connection method is the following.

Let us consider a particle moving freely under purely gravitational force. According to the equivalence principle there exist a freely falling coordinate system ξ^α in which the motion of the particle is rectilinear and is described by the following equation:

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{6}$$

where $d\tau$ is the proper time

$$d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta, \qquad \eta_{\alpha\beta} = \begin{cases} \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1 \\ 0, \qquad \alpha \neq \beta \end{cases} \tag{7}$$

We now suppose that at our disposal there is any other coordinate system x^ν which may be regarded as resting Cartesian coordinate with respect to the laboratory one and also as curvilinear, accelerated, rotating, or any other coordinate system we wish. Coordinates ξ^α of the freely falling coordinate system are functions of x^μ and equation (6) takes the form

$$\frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial \xi^\alpha}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

Multiplying this equation by $\partial x^{\lambda}/\partial\xi^{\alpha}$ and using the well-known rule of multiplication,

$$\frac{\partial\xi^{\alpha}}{\partial x^{\nu}}\frac{\partial x^{\lambda}}{\partial\xi^{\alpha}}=\delta^{\lambda}_{\nu}$$

we have the following equation for the particle motion

$$\frac{d^2x^{\lambda}}{d\tau^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=0 \tag{8}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is called the affine connection and is determined by the relation

$$\Gamma^{\lambda}_{\mu\nu}\equiv\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}\equiv\frac{\partial x^{\lambda}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial x^{\mu}\,\partial x^{\nu}} \tag{9}$$

The proper time (7) may also be expressed in an arbitrary coordinate system

$$d\tau^2=\eta_{\alpha\beta}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}dx^{\mu}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}dx^{\nu}$$

or

$$d\tau^2=g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}$$

where $g_{\mu\nu}$ is the metric tensor which is given by

$$g_{\mu\nu}\equiv\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta}$$

It turns out that the affine connection method can be applied to define geometrical and physical objects in quantized space-time with coordinates (2). Thus, in our consideration there are three coordinate systems: inertial (with the coordinates ξ^{α} and the metric tensor $\eta_{\alpha\beta}$), noninertial (with the coordinates x^{ν} and the metric tensor $g_{\mu\nu}$), and general [with the quantized coordinates $\hat{x}^{\nu}$ (2) and corresponding metric tensor is denoted $g_{\mu\nu}(\hat{x})$]. Then, from the constancy principle of the proper time in any coordinate system, we have

$$d\tau^2=\eta_{\alpha\beta}\,d\xi^{\alpha}\,d\xi^{\beta}=g_{\mu\nu}(x)\,dx^{\mu}\,dx^{\nu}=g_{\mu\nu}(\hat{x})\,d\hat{x}^{\mu}\wedge d\hat{x}^{\nu} \tag{10}$$

where the symbol $\wedge$ means noncommutativity of the multiplication law of variables $\hat{x}^{\nu}$.

We know the affine connection between coordinates ξ^{α} and x^{ν}, which is given by (9). Similar calculation may be made for the passage from coordinate system x^{ν} to the quantized one $\hat{x}^{\mu}$, the result reads

$$\Gamma^{\lambda}_{\nu\mu}(\hat{x})=\frac{\partial\hat{x}^{\lambda}}{\partial x^{\delta}}\frac{\partial x^{\alpha}}{\partial\hat{x}^{\nu}}\wedge\frac{\partial x^{\beta}}{\partial\hat{x}^{\mu}}\{\Gamma^{\delta}_{\alpha\beta}\}+\frac{\partial\hat{x}^{\lambda}}{\partial x^{\alpha}}\frac{\partial^2x^{\alpha}}{\partial\hat{x}^{\nu}\wedge\partial\hat{x}^{\mu}} \tag{11}$$

In order to obtain the explicit form of the Christoffel symbol $\Gamma^{\lambda}_{\nu\mu}(\hat{x})$ we must define the concrete dependence of quantized coordinates $\hat{x}^{\mu}$ from x^{μ}, i.e., form of $\Gamma^{\mu}(x)$ in (2). We give this dependence within the framework of the tetrad formalism in which we assume

$$\Gamma^{\mu}(x) = \Gamma^{a}\, e^{\mu}_{a}(x) \tag{12}$$

where Γ^{a} are matrices (of the type of Dirac matrices γ^{a}) with zero traces or their different combinations, independent of the coordinates x^{μ} and satisfying the following relation:

$$\Gamma^{a}\Gamma^{b} + \Gamma^{b}\Gamma^{a} = 2\eta^{ab}$$

[η^{ab} is the metric tensor of the inertial coordinate system, diag $\eta = (1, -1, -1, -1)$]. e^{μ}_{a} is the "vielbein" field which specifies the basic vectors of the linear tangent space at each point of a curved space-time manifold. This implies that e^{μ}_{a} is nonsingular, and has an inverse e^{a}_{μ}, defined by

$$e^{\mu}_{a}\, e^{b}_{\mu} = \delta^{b}_{a}, \qquad e^{a}_{\mu}\, e^{\nu}_{a} = \delta^{\nu}_{\mu}$$

The standard nomenclature is to call e^{a}_{μ} the vielbein and e^{μ}_{a} the inverse vielbein. Here through $\mu, \nu, \ldots$ and $a, b, \ldots$ we denote world indices and tetrad indices (or local Lorentz indices), respectively. From (2), after straightforward calculation, we have

$$d\hat{x}^{\mu} = dx^{\mu} + l\Gamma^{a} \frac{\partial e^{\mu}_{a}(x)}{\partial x^{\nu}}\, dx^{\nu}$$

or

$$\frac{d\hat{x}^{\mu}}{dx^{\nu}} = \delta^{\mu}_{\nu} + l\Gamma^{a} \frac{\partial e^{\mu}_{a}(x)}{\partial x^{\nu}} \tag{13}$$

Here we have assumed that $\hat{x}$ and x spaces are independent of each other and therefore the following standard relations hold:

$$\frac{\partial \hat{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \hat{x}^{\mu}} = \delta^{\nu}_{\mu}$$

or

$$\frac{\partial \hat{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \hat{x}^{\nu}} = \delta^{\beta}_{\alpha} \tag{14}$$

On the other hand, making use of (14) we get from (2)

$$\frac{\partial x^{\alpha}}{\partial \hat{x}^{\nu}} = \delta^{\alpha}_{\nu} - l\Gamma^{a} \frac{\partial e^{\alpha}_{a}}{\partial x^{\nu}} + l^{2}\Gamma^{a}\Gamma^{b} \frac{\partial e^{\alpha}_{a}}{\partial x^{\mu}} \frac{\partial e^{\mu}_{b}}{\partial x^{\nu}} - \cdots \tag{15}$$

Thus, by using the formulas (11), (13), and (15) one can define $\Gamma^{\lambda}_{\nu\mu}(\hat{x})$ the explicit form of which will be given below in Section 4.

The vielbein field $e^{\mu}_{a}(x)$ is the world vector with respect to the index μ and its transformation law under the passage from one to another coordinate system is defined by the standard way, i.e.,

$$e^{\nu}_{a}(\hat{x}) = \frac{\partial \hat{x}^{\nu}}{\partial x^{\beta}} e^{\beta}_{a}(x)$$

or

$$e^{\nu}_{a}(\hat{x}) = e^{\nu}_{a}(x) + l\Gamma^{b} e^{\beta}_{a}(x) \frac{\partial e^{\nu}_{b}}{\partial x^{\beta}} \tag{16}$$

The existence of the inverse vielbeins allows the introduction of a covariant metric tensor

$$g^{\alpha\beta}(x) = e^{\alpha}_{a}(x) e^{\beta}_{b}(x) \eta^{ab} \tag{17}$$

where η^{ab} is a Lorentz invariant tensor, which can be used for local measurements of distances and angles in space-time. On the other hand, from (10) we get

$$g^{\nu\mu}(\hat{x}) = \frac{\partial \hat{x}^{\nu}}{\partial x^{\alpha}} \wedge \frac{\partial \hat{x}^{\mu}}{\partial x^{\beta}} g^{\alpha\beta}(x) \tag{18}$$

Taking into account (13) and (17) we have finally

$$\begin{aligned} g^{\nu\mu}(\hat{x}) = g^{\nu\mu}(x) + l\left[g^{\alpha\nu}(x) \frac{\partial \Gamma^{\mu}(x)}{\partial x^{\alpha}} + g^{\alpha\mu}(x) \frac{\partial \Gamma^{\nu}(x)}{\partial x^{\alpha}} \right] \\ + l^{2} g^{\alpha\beta}(x) \frac{\partial \Gamma^{\nu}(x)}{\partial x^{\alpha}} \wedge \frac{\partial \Gamma^{\mu}(x)}{\partial x^{\beta}} \end{aligned} \tag{19}$$

From this we see that in our model the metric tensor consists of two parts: symmetrical-usual classical and additional-quantized ones with antisymmetrical term due to quantized space-time. The latter part characterizes just the quantum property of space-time at small distances and disappears in the limit $l \to 0$. In the language of the particle structure this part may be connected with internal quantum properties of the particle such as spin, isospin, electric and magnetic charges, etc. A majority of physicists believe that metric tensor is symmetric. However, in light of recent development of the unified gauge theory of all interactions including gravity in which geometrical aspects (topology, anomalies related with string and superstring theories), astrophysical problems (torsion, rotation, and asymmetry of the universe), and exotic phenomena such as magnetic monopoles and black holes, etc. are discussed intensively, it is very important to study structure of space-time with the metric tensor of both types. In our opinion, quantum property of space-time is crucial in the solution of the above-mentioned problems and may give important new ways of understanding physical phenomena at a deeper level. Here we show that the essence of the torsion and the existence of the magnetic monopole are caused by the quantum nature of space-time at small distances. The last case may also be understood in another manner. One can say that if there exist magnetic monopoles, the space-time structure in the surroundings of them should always be quantized.

Before going into these problems we now attempt to link quantized space-time property with the spin structure of the particle. For this, we consider the local field $\varphi(x)$ and general coordinate transformations, i.e., arbitrary reparametrizations of space-time

$$x^\mu \Rightarrow x^\mu + \xi^\mu(x) \tag{20}$$

Since spin is an intrinsic aspect of any theory one must also consider local rotations of spin, which act on the field $\varphi(x)$ according to

$$\varphi(x) \Rightarrow \varphi(x) + \tfrac{1}{2}\varepsilon^{ab}(x)\Sigma_{ab}\varphi(x)$$

where Σ_{ab} are the generators of the spin rotation group G in arbitrary dimensions in the representation appropriate to the field $\varphi(x)$. If we restrict ourselves to the Lorentz group, so that the generators Σ_{ab} are antisymmetric in a and b and satisfy the commutator algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = -\delta_{ac}\Sigma_{bd} + \delta_{ad}\Sigma_{bc} + \delta_{bc}\Sigma_{ad} - \delta_{bd}\Sigma_{ac} \tag{21}$$

As usual (see, for detail, de Wit, 1984), at the same time of spin rotation, the concept of the spin connection is introduced, which appears in the covariant derivative

$$D_\mu\varphi = \partial_\mu\varphi - \tfrac{1}{2}\omega_\mu^{ab}\Sigma_{ab}\varphi$$

and transforms under spin rotation in the standard fashion,

$$\delta\omega_\mu^{ab} = \partial_\mu\varepsilon^{ab} - \omega_{\mu c}^{\ a}\varepsilon^{cb} - \omega_{\mu c}^{\ b}\varepsilon^{ac}$$

where we made use of (21). The presence of a world index in ω_μ^{ab} is characteristic for gauge fields, therefore we call it the gauge field or the spin connection field. It is shown (see de Wit, 1984) that in the local tangent space frame determined by e_a^μ the spin connection field ω_μ^{ab} may be expressed by means of the given vielbein e_a^μ:

$$\omega_\mu^{ab} \Rightarrow \omega_\mu^{ab}(e) = \tfrac{1}{2}e_\mu^c(\Omega_{ab}^{\ \ c} - \Omega_{bc}^{\ \ a} - \Omega_{ca}^{\ \ b}) \tag{22}$$

where $\Omega_{ab}^{\ \ c}$ is the object of anholonomity which measures the noncommutativity of the vielbein basis

$$\Omega_{ab}^{\ \ c} = e_a^\mu e_b^\nu(\partial_\mu e_\nu^c - \partial_\nu e_\mu^c) \tag{23}$$

On the other hand, if introducing the method of quantized space-time will be regarded as coordinate transformations as in equation (20), then after standard calculations of the variation for the vielbein field e^a_μ, we obtain

$$\delta e^a_\nu = l\Gamma^\mu(\partial_\nu e^a_\mu - \partial_\mu e^a_\nu) \tag{24}$$

Comparing this value with (22) and (23) one can say that introduction of quantized space-time by formula (2) may be connected with internal spin structure of the particle. As is shown by de Wit (1984), if spin rotation given by (22) and (23) is carried out, then the general affine connection will become nonsymmetric and it in turn gives rise to the torsion tensor. An analogous situation exists in our case.

4. THE TORSION TENSOR

First, consider the affine connection given by (11). Taking into account (13) and (15), and keeping terms of the order of l^2, we have after simple calculations

$$\begin{aligned}\Gamma^\lambda_{\nu\mu}(\hat{x}) = \left\{ {\lambda \atop \nu\mu} \right\} &+ l\left[\frac{\partial\Gamma^\lambda}{\partial x^\rho}\left\{ {\rho \atop \nu\mu} \right\} - \frac{\partial\Gamma^\delta}{\partial x^\nu}\left\{ {\lambda \atop \delta\mu} \right\} - \frac{\partial\Gamma^\delta}{\partial x^\mu}\left\{ {\lambda \atop \delta\nu} \right\} - \frac{\partial^2\Gamma^\lambda}{\partial x^\nu\,\partial x^\mu}\right] \\ &+ l^2\left[\frac{\partial\Gamma^\sigma}{\partial x^\nu}\frac{\partial\Gamma^\delta}{\partial x^\mu}\left\{ {\lambda \atop \sigma\delta} \right\} + \frac{\partial\Gamma^\sigma}{\partial x^\alpha}\frac{\partial\Gamma^\alpha}{\partial x^\nu}\left\{ {\lambda \atop \sigma\mu} \right\}\right. \\ &+ \frac{\partial\Gamma^\sigma}{\partial x^\alpha}\frac{\partial\Gamma^\alpha}{\partial x^\mu}\left\{ {\lambda \atop \sigma\nu} \right\} + \frac{\partial^2\Gamma^\lambda}{\partial x^\alpha\,\partial x^\mu}\frac{\partial\Gamma^\alpha}{\partial x^\nu} + \frac{\partial^2\Gamma^\lambda}{\partial x^\beta\,\partial x^\nu}\frac{\partial\Gamma^\beta}{\partial x^\mu} \\ &\left. - \frac{\partial\Gamma^\lambda}{\partial x^\rho}\frac{\partial\Gamma^\sigma}{\partial x^\nu}\left\{ {\rho \atop \mu\sigma} \right\} - \frac{\partial\Gamma^\lambda}{\partial x^\rho}\frac{\partial\Gamma^\sigma}{\partial x^\mu}\left\{ {\rho \atop \nu\sigma} \right\}\right]\end{aligned}$$

From this expression we see that the affine connection contains such terms which are antisymmetric with respect to the rearrangement of the indices ν, μ. It is well known that the antisymmetric part of the affine connection determines the torsion tensor (see de Wit, 1984), i.e.,

$$\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e^\lambda_a Q^a_{\mu\nu}$$

where

$$Q^a_{\mu\nu} = l^2 \frac{\partial e^a_\sigma}{\partial x^\delta}\frac{\partial e^\sigma_c}{\partial x^\nu}\frac{\partial e^\delta_b}{\partial x^\mu}[\Gamma^c\Gamma^b - \Gamma^b\Gamma^c] \tag{25}$$

arises from the antisymmetric part of the metric tensor and is called the torsion tensor, the physical meaning of which is connected to internal rotations of space-time itself. It is obvious that in the classical limit $l \to 0$ both the antisymmetric part of the metric tensor and the torsion tensor become zero. This means that the torsion tensor is indeed connected with the quantum property of space-time at small distances and its value is very small of the order of l^2.

Let us consider the particular case, when $\Gamma^a = i\gamma^a\gamma^5$, γ^a are the usual Dirac matrices. At this the torsion tensor (25) takes the form

$$Q^a_{\mu\nu} = \frac{l^2\,\partial e^a_\sigma}{\partial x^\delta}\frac{\partial e^\sigma_c}{\partial x^\nu}\frac{\partial e^\delta_b}{\partial x^\mu}\Sigma^{cb} \tag{26}$$

where

$$\Sigma^{cb} = \gamma^c\gamma^b - \gamma^b\gamma^c$$

is associated with the generator of the spin rotation in tetrad formulation.

Recently, torsion and related problems have been the subject of many papers (see, for example, Ivanenko, 1983; Obukhov, 1983; Trautman, 1980). In particular, torsion can prevent collapse. It also leads to an interesting precession of spin in the space endowed with torsion which could provide a method for measuring torsion. One of the most important consequences of torsion is its induction of nonlinearities, e.g., in Dirac's equation—of cubic pseudovector type, having the form proposed earlier by Ivanenko (1938):

$$[\gamma^a\partial_a + m + \lambda(\bar\psi\gamma^a\gamma^5\psi)\gamma^a\gamma^5]\psi = 0$$

It is no exception that the torsion over the whole scale of the universe may give rise to its rotation with the angular velocity of the order of 10^{-13} rad/yr indicated by Birch (1982).

Now we define the curvature of space-time within the framework of our formalism. Since the metric tensor and the affine connection have additional terms due to quantized space-time structure and therefore the curvature is also changed in our case. By using the usual definition of the curvature and carrying out some calculations and keeping terms of the order of l in the obtained expression for the generic curvature, we obtain

$$r^\lambda_{\rho,\nu\mu}(\hat{x}) = R^\lambda_{\rho,\nu\mu}(x) + l\Gamma^a\frac{\partial e^\lambda_a}{\partial x^\sigma}R^\sigma_{\rho,\nu\mu} - l\Gamma^a\frac{\partial e^\sigma_a}{\partial x^\rho}R^\lambda_{\sigma,\nu\mu}$$

$$- l\Gamma^a\frac{\partial e^\sigma_a}{\partial x^\nu}R^\lambda_{\rho,\sigma\mu} - l\Gamma^a\frac{\partial e^\sigma_a}{\partial x^\mu}R^\lambda_{\rho,\nu\sigma}$$

where $R^\lambda_{\rho,\nu\mu}(x)$ is the usual Einstein curvature tensor depending on the classical coordinates x^ν. It is very interesting to carry out an investigation of Einstein's equation constructed by means of this new curvature. However, this problem is beyond the scope of this paper and is a matter of our further work. Here our next step is to study the problem of magnetic monopole from the quantized space-time point of view.

5. DIRAC MAGNETIC MONOPOLES IN QUANTIZED SPACE-TIME

The possible existence of magnetic monopoles in nature is a very exotic problem of physics during its history. One of the earliest recorded discussion of magnetic monopoles is found in a latter written by Petrus Peregrinus de Maricourt in AD 1269, it was addressed to the great magnetist Gilbert and contains the initial idea of poles and lines of force. Maxwell also considered magnetic poles in his unification of electricity and magnetism but lack of experimental evidence caused them to abstain from its final formulation. J. J. Thomson (1904) had noticed the remarkable theoretical fact that the electromagnetic angular momentum in a magnetic pole–electric charge system was independent of their separation. The great modern physicist Dirac (1931) studied the magnetic monopole problem within the framework of quantum mechanics and special relativity. Dirac demonstrated that a single magnetic pole anywhere in the universe would explain the fact that all electric charge occurs only as discrete integral multiples of e, the charge of the electron, in essence electric charge quantization

$$eg = n(\hbar c/2)$$

where e and g are the electric and magnetic charges, respectively, and n is the principal quantum number. The monopole charge, if there exists one, is 70 times larger than the electric charge. This fact gives rise to two consequences playing an important role in the experimental search of magnetic monopoles: a rapidly moving monopole should produce heavy ionization as it passes through matter and monopoles should bind to some forms of matter. In the Dirac theory other monopole properties: size, shape, mass, parity, spin, etc. are arbitrary and, moreover, some mathematical problem appears. The point magnetic charges of Dirac monopoles are singular, in the sense that they are singularities of the electromagnetic field, i.e., the vector potential A_μ has a line singularity, and the "Dirac veto" or the Dirac string, which leads to the quantization, is the requirement that the electron wave function vanish on this line-string. Wu and Yang (1975) have reformulated Dirac's theory to avoid any singularities in A_μ. It turns out that the mathematical structure of the Wu–Yang theory is that of fiber bundles and the Dirac monopole does also have a topological origin (see, for example, Ryder, 1977).

The next important step in the theory of magnetic monopoles was taken by Polyakov (1974) and t' Hooft (1974) within the gauge theory of particle interactions. They independently showed that monopoles appear as stable solutions of spontaneously broken Yang–Mills field equations and are required by a large class of theories (see, for example, Goddard and Olive, 1978, and Coleman, 1982). If this gauge theory is correct, 't Hooft–Polyakov monopoles inevitably exist, and their mass is not arbitrary, at least, of the order of $\sim 100m_{\mathrm{W}}$ or larger ($m_{\mathrm{W}} \sim 100$ GeV is the intermediate vector boson mass of the electroweak theory due to Weinberg, 1967; Salam, 1968; and Glashow, 1961) and they are of finite size. Owing to the non-Abelian structure of the gauge group no string singularity appears, in contrast to the Abelian theory of monopoles, developed originally by Dirac (1931), where the string was required in order to preserve the magnetic flux. Another peculiarity of 't Hooft–Polyakov monopoles is that magnetic charge has a topological origin; that is to say that the boundary conditions on the fields are ones which cannot be changed continuously in constant values. The asymptotic field configuration is topologically nontrivial, and this gives rise to the quantized magnetic charge.

Magnetic monopoles also do appear in the so-called grand unification theories, GUT's, that unify strong and electroweak forces. Rapid quenching of the Higgs fields in the universe expanding process leads to topological defects (Kibble, 1976) which are magnetic monopoles whose masses are typically $\sim 10^{16}$ GeV.

Recently, many attempts were made by experimentalists (see, review due to Carrigan and Trower, 1982; 1983) to detect magnetic monopoles, but they have up to now not seen them. However, the necessity of their existence is dedicated to the mathematical beauty of the physical theory based on the symmetry principle of nature.

Now we go on to discuss the magnetic monopole problem from the quantized space-time point of view. Our aim is modest. We show that if space-time is quantized the magnetic monopole should exist as a direct consequence of the geometrical structure of this space-time. Generally speaking, in the usual classical space there is no place for magnetic monopoles described by regular potential $\mathbf{A}(\mathbf{x}, t)$. This assertion is based on the following simple geometrical fact. Let $\mathbf{A}(\mathbf{x})$ be some regular static magnetic potential given the magnetic field $\mathbf{H} = \mathrm{rot}\,\mathbf{A}$. On the other hand, if $\mathbf{H}$ is indeed real observable field caused by a point magnetic charge g, then it should give a magnetic flux through some closed surface S containing the charge g and bounded the volume V:

$$P = \oint_S (\mathbf{H} \cdot d\mathbf{S}) = \int_V dV\, \mathrm{div}\,\mathrm{rot}\,\mathbf{A} = 4\pi g \tag{27}$$

However, because of the geometrical structure of classical space

$$\operatorname{div} \operatorname{rot} \mathbf{A} = \varepsilon^{ijk} \frac{\partial^2}{\partial x^i \, \partial x^j} A_k \equiv 0 \qquad (i, j, k = 1, 2, 3)$$

It is obvious that in order to coordinate these two contradictory facts it should be either assumed that $\mathbf{A}$ possesses some singularities or the topological structure of space in which $\operatorname{div} \operatorname{rot} \mathbf{A} \neq 0$ should be changed for any regular value of $\mathbf{A}$. It turns out that owing to noncommutability of the quantized coordinates $[\hat{x}^i, \hat{x}^j] \neq 0$, $\operatorname{div} \operatorname{rot} \mathbf{A}(\hat{\mathbf{x}})$ is not vanishing in our scheme and takes the form

$$\operatorname{div} \mathbf{H}(\hat{\mathbf{x}}) = \varepsilon^{ijk} \frac{\partial^2}{\partial \hat{x}^i \wedge \partial \hat{x}^j} A_k(\hat{\mathbf{x}}) \tag{28}$$

where $\partial \hat{x}^i \wedge \partial x^j = -\partial x^j \wedge \partial x^i$ which follows from definition (2). Taking into account this definition and after some calculations we have

$$\operatorname{div} \mathbf{H}(\hat{\mathbf{x}}) = \varepsilon^{ijk} \frac{\partial^2 A_k(\mathbf{x})}{\partial x^i \, \partial x^j} + l^2 \varepsilon^{ijk} \frac{\partial^2 A_k(\mathbf{x})}{\partial x^m \, \partial x^n} \frac{\partial \Gamma^n(\mathbf{x})}{\partial x^i} \wedge \frac{\partial \Gamma^m(\mathbf{x})}{\partial x^j} \tag{29}$$

where we recall that $\Gamma^n(\mathbf{x}) = \Gamma^a e_a^n(\mathbf{x})$, $e_a^n(x)$ are the tetrad fields $(i, j, k, n, m = 1, 2, 3)$. The first term of (29) coincides with the usual classical expression for $\operatorname{div} \operatorname{rot} \mathbf{A}(\mathbf{x})$ (it goes to zero) and the second term appears due to quantized space-time. In (29) we also have not written terms which are symmetric over indices i, j and turn to zero after using the fact that ε^{ijk} is a fully antisymmetric tensor of the third rank. In order to calculate the explicit form of (29) we concretize Γ^a matrices; $\Gamma^a = i\gamma^a\gamma^5$ and passage to the nonrelativistic limit for the given case, and use the following relation:

$$\Gamma^a\Gamma^b - \Gamma^b\Gamma^a \Rightarrow \sigma^a\sigma^b - \sigma^b\sigma^a = 2i\varepsilon^{abc}\sigma^c$$

where σ^a are the Pauli matrices. Further, using the identity

$$\varepsilon^{ijk} \frac{\partial \Gamma^n(\mathbf{x})}{\partial x^i} \wedge \frac{\partial \Gamma^m(\mathbf{x})}{\partial x^j} = \frac{1}{2} \varepsilon^{ijk} (\Gamma^a\Gamma^b - \Gamma^b\Gamma^a) \frac{\partial e_a^n}{\partial x^i} \frac{\partial e_b^m}{\partial x^j}$$

$$\Rightarrow i\varepsilon^{ijk} \varepsilon^{abc} \sigma^c \frac{\partial e_a^n}{\partial x^i} \frac{\partial e_b^m}{\partial x^j}$$

we have for (29)

$$\operatorname{div} \mathbf{H}(\hat{\mathbf{x}}) = i l^2 \varepsilon^{ijk} \varepsilon^{abc} \frac{\partial^2 A_k(\mathbf{x})}{\partial x^m \, \partial x^n} \sigma^c \frac{\partial e_a^n}{\partial x^i} \frac{\partial e_b^m}{\partial x^j} \tag{30}$$

Now we assume that the magnetic potential $A_k(\mathbf{x})$ is regular everywhere and find its value by using the requirement that the averaged magnetic flux of (30) equals $4\pi g$

$$\langle P\rangle = \int d^3x \langle \operatorname{div} \mathbf{H}(\hat{\mathbf{x}})\rangle = 4\pi g \tag{31}$$

where the averaging procedure leads to taking the trace of (30). First of all, from (30) and (31) we see that $A_k(\mathbf{x}) \sim \sigma^k f(\mathbf{x})$. Further, as usual, we assume that in our large-scale nonquantum space-time there is no selected direction and therefore the value of $f(\mathbf{x})$ must depend on the distance $r = (x^2+y^2+z^2)^{1/2}$ only. Thus, we suppose that

$$A_k(\mathbf{x}) = \begin{cases} 0 & \text{for } r \le l \\ \dfrac{\sigma^k}{i}\, g f(r) & \text{for } r > l \end{cases} \tag{32}$$

and determine an explicit form of $f(r)$, satisfying expression (31). The assumption that $A_k(\mathbf{x}) = 0$, for $r \le l$ is based on the regularity proposal about $A_k(\mathbf{x})$ at the point $\mathbf{x} = 0$. According to the assumption $A_k(\mathbf{x}) = -ig\sigma^k f(\mathbf{r})$, the expression (30) takes the form

$$\langle \operatorname{div} \operatorname{rot} \mathbf{A}(\mathbf{x})\rangle = g\frac{l^2}{2}\operatorname{Sp}(\sigma^c\sigma^k)\left[x^n x^m\left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right] + f'(r)\frac{\delta_{nm}}{r}\right]$$
$$\times\, \varepsilon^{ijk}\varepsilon^{abc}\frac{\partial e_a^n}{\partial x^i}\frac{\partial e_b^m}{\partial x^j} \tag{33}$$

Here the terms with δ_{nm} turn to zero and according to (3) the normalizing factor $\frac{1}{2}$ arises from the fact that σ^k is a two-column matrix. Taking the trace and using the identity

$$\varepsilon^{abk}\varepsilon^{ijk} = \delta_{ai}\,\delta_{bj} - \delta_{aj}\,\delta_{bi}$$

we have finally

$$\langle \operatorname{div} \operatorname{rot} \mathbf{A}(\mathbf{x})\rangle = g l^2\left[\frac{f''(r)}{r^2} - \frac{f'}{r^3}\right] I \tag{34}$$

where

$$I = x^n x^m\left(\frac{\partial e_a^n}{\partial x^a}\frac{\partial e_b^m}{\partial x^b} - \frac{\partial e_a^n}{\partial x^b}\frac{\partial e_b^m}{\partial x^a}\right)$$

In the definition of the magnetic flux (31) the term with I enters into the integral over the polar variables and remaining integral over the radial variable r should be

$$\int dr\cdot r^2[f''(r)/r^2-f'(r)/r^3]=\text{const} \tag{35a}$$

From this condition, it is easy to find an explicit form of $f(r)$. Indeed, using the identity

$$d\left(f'+\frac{f}{r}\right)=f''\,dr+\frac{f'}{r}\,dr-\frac{f}{r^2}\,dr$$

or

$$\frac{f'}{r}\,dr=d\left(f'+\frac{f}{r}\right)-f''\,dr+\frac{f}{r^2}\,dr$$

we obtain

$$\int dr(f''-f'/r)=2\int drf''(r)-\int d\left(f'+\frac{f}{r}\right)-\int\frac{f}{r^2}\,dr \tag{35b}$$

On the other hand, making use of partial integration in (35a) we get

$$\int dr(f''-f'/r)=\int dr\,f''-f\frac{1}{r}\bigg|_{\text{boun.}}-\int\frac{f}{r^2}\,dr \tag{36}$$

The expressions (35b) and (36) should be equal to each other for any boundary condition:

$$\int^r d\left[f'(x)+\frac{f(x)}{x}\right]=\left(f'+\frac{f}{r}\right)\bigg|_{\text{boun.}}$$

From the requirement that the magnetic flux does not depend on the value of the integration limits (in particular, $0\le r<\infty$ or $l\le r<\infty$), we have the following differential equation for $f(r)$:

$$f'(r)+f(r)/r=0$$

the solution of which is simple and has the form

$$f(r)=c/r$$

with some constant c. Assuming $c=1$, we finally get

$$\mathbf{A}(\mathbf{x})=\begin{cases}0, & \text{for } r\le l\\ \dfrac{\boldsymbol{\sigma}}{i}\,g\,\dfrac{1}{r}, & \text{for } r>l\end{cases} \tag{37}$$

It should be noted that instead of the requirement $\mathbf{A}(\mathbf{x})=0$ for $r \le l$ in (37), of course, it is possible any other cases depending on an inner structure of the magnetic monopole, for example, $A(x)=gr/l^2$ or gr^2/l^3, for $r \le l$.

Now we show that by this choice of the magnetic potential (37) its flux satisfies relation (31). For this, a concrete form of the tetrad field e_μ^n should be defined. For simplicity, we choose the spherical frame of reference as the tetrad coordinate system and the Cartesian one for the world coordinate system. Then the tetrad field e_μ^a has the form

$$e_\mu^a = \partial\xi^a/\partial x^\mu, \qquad e_a^\mu = \partial x^\mu/\partial\xi^a$$

where

$$d\xi^1 = dr, \qquad d\xi^2 = r\,d\theta, \qquad d\xi^3 = \rho\,d\varphi$$
$$dx^1 = dx, \qquad dx^2 = dy, \qquad dy^3 = dz, \qquad dx^0 = c\,dt$$

and $r=(x^2+y^2+z^2)^{1/2}$, $\rho=(x^2+y^2)^{1/2}$. One can easily see that the field e_μ^a is given by

$$e_\mu^a = \begin{pmatrix} x/r & zx/r\rho & -y/\rho & 0 \\ y/r & zy/r\rho & x/\rho & 0 \\ z/r & -\rho/r & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

Taking into account (38) and carrying out the necessary calculation, we get from (34) with (37)

$$\langle P\rangle = \oint_S (\mathbf{H}\,d\mathbf{S}) = \int d^3x \langle \operatorname{div}\mathbf{H}(\mathbf{x})\rangle = 3l^2 g \frac{8}{3}\pi \int_l^\infty \frac{r^2\,dr}{r^5} = 4\pi g \tag{39}$$

where

$$\int d\Omega\, I = \tfrac{8}{3}\pi$$

Thus, we arrive at the relation (31). From our result may be concluded that if space-time structure possesses quantized nature then in it there is place for magnetic monopoles or, in contrary, if there exist magnetic monopoles then space-time surroundings of them should have a quantized property.

In conclusion, it should be noted that magnetic potential in quantized space-time is regular everywhere and may be defined from the physical condition imposed on the value of the magnetic flux which equals $4\pi g$.

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